Class 12 - Chapter 1: Relations and Functions

Complete Revision Notes for CBSE & RBSE Board Exams

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1. Introduction

Relations and Functions are fundamental concepts in mathematics that describe connections between elements of sets. This chapter covers:

- **Relations:** Definition, types, and equivalence relations
- **Functions:** One-one, onto, bijective, and invertible functions
- **Composition:** Combining functions and their properties
- **Applications:** Real-world problem-solving using these concepts

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2. Relations: Definition and Types

What is a Relation?

A **relation R from set A to set B** is any subset of $A \times B$.

If $(a, b) \in R$, we say "a is related to b" under relation R, written as aRb.

Empty and Universal Relations

Empty Relation: $R = \emptyset \subset A \times A$ (no elements are related)

Universal Relation: $R = A \times A$ (all elements are related)

Both are called **trivial relations**.

Example

For $A = \{1, 2, 3, 4\}$:

- $R_1 = \{(a, b) : a - b = 10\}$ is empty (no pairs satisfy condition)
- $R_2 = \{(a, b) : |a - b| \geq 0\}$ is universal (all pairs satisfy condition)

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3. Important Properties of Relations

Definition 3: Three Key Properties

Reflexive: A relation R in A is reflexive if $(a, a) \in R$ for every $a \in A$

Symmetric: A relation R in A is symmetric if $(a_1, a_2) \in R$ implies $(a_2, a_1) \in R$

Transitive: A relation R in A is transitive if $(a_1, a_2) \in R$ and $(a_2, a_3) \in R$ implies $(a_1, a_3) \in R$

Summary Table

Relation Type	Reflexive	Symmetric	Transitive
Empty Relation	✗	✓	✓
Universal Relation	✓	✓	✓
Identity Relation	✓	✓	✓
Less than ($<$)	✗	✗	✓
Perpendicular Lines	✗	✓	✗
Congruent Triangles	✓	✓	✓

Table 1: Properties of Common Relations

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4. Equivalence Relations

Definition 4: Equivalence Relation

A relation R in set A is an **equivalence relation** if it is simultaneously:

1. **Reflexive** - $(a, a) \in R$ for all $a \in A$
2. **Symmetric** - $(a, b) \in R \Rightarrow (b, a) \in R$
3. **Transitive** - $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$

Example: Congruence of Triangles

$R = \{(T_1, T_2) : T_1 \text{ is congruent to } T_2\}$ is an equivalence relation because:

- **Reflexive:** Every triangle is congruent to itself
- **Symmetric:** If $T_1 \cong T_2$, then $T_2 \cong T_1$
- **Transitive:** If $T_1 \cong T_2$ and $T_2 \cong T_3$, then $T_1 \cong T_3$

Example: Divisibility by 2

$R = \{(a, b) : 2 \text{ divides } (a - b)\}$ in set $\mathbb{Z}$ is equivalence:

- **Reflexive:** 2 divides $(a - a) = 0$ ✓
- **Symmetric:** If 2|(a-b), then 2|(b-a) ✓
- **Transitive:** If 2|(a-b) and 2|(b-c), then 2|[(a-b)+(b-c)] = 2|(a-c) ✓

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5. Equivalence Classes

Definition: Equivalence Class

For an equivalence relation R in set A and element $a \in A$, the **equivalence class** $[a]$ is:

$$[a] = \{b \in A : (a, b) \in R\}$$

(Set of all elements related to a)

Key Properties

For equivalence relation R in set A:

1. **All elements within a class are related to each other**
2. **No element of one class relates to elements of another class**
3. **Classes partition the set:** $A = [a_1] \cup [a_2] \cup \ldots$ and $[a_i] \cap [a_j] = \emptyset$ for $i \neq j$

Example: Even and Odd Integers

For $R = \{(a, b) : 2|(a - b)\}$ in $\mathbb{Z}$:

- $[0] = \{\text{all even integers}\} = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$
- $[1] = \{\text{all odd integers}\} = \{\ldots, -3, -1, 1, 3, 5, \ldots\}$

Note: $[0] = [2] = [4] = \ldots$ and $[1] = [3] = [5] = \ldots$

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6. Types of Functions

Definition 5: One-One (Injective) Function

Function $f : X \to Y$ is **one-one** if distinct elements of X map to distinct elements of Y:

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \text{ for all } x_1, x_2 \in X$$

Contrapositive: If $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$

Definition 6: Onto (Surjective) Function

Function $f : X \to Y$ is **onto** if every element of Y is the image of some element in X:

$$\text{For every } y \in Y, \exists x \in X \text{ such that } f(x) = y$$

Key: Range of f = Codomain = Y

Definition 7: One-One and Onto (Bijective) Function

Function $f : X \to Y$ is **bijective** (one-one and onto) if it is both:

- One-one (injective)
- Onto (surjective)

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7. Examples of Function Types

Example 1: $f : \mathbb{N} \to \mathbb{N}$, $f(x) = 2x$

One-one? Yes - $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$ ✓

Onto? No - 1 has no preimage (no x with $2x = 1$) ✗

Conclusion: One-one but NOT onto

Example 2: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x$

One-one? Yes - $2x_1 = 2x_2 \Rightarrow x_1 = x_2$ ✓

Onto? Yes - For any $y \in \mathbb{R}$, $x = \frac{y}{2}$ satisfies $f(x) = y$ ✓

Conclusion: Bijective (one-one and onto)

Example 3: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

One-one? No - $f(-1) = 1 = f(1)$ but $-1 \neq 1$ ✗

Onto? No - $-2 \in \mathbb{R}$ (codomain) has no preimage ✗

Conclusion: Neither one-one nor onto

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8. Composition of Functions

Definition 8: Composition of Functions

For functions $f : A \to B$ and $g : B \to C$, the **composition** $g \circ f : A \to C$ is:

$$(\text{g}\circ\text{f})(\text{x}) = \text{g}(\text{f}(\text{x})) \text{ for all } x \in A$$

Key: Apply f first, then apply g to the result.

Important Property: Non-Commutativity

In general: $g \circ f \neq f \circ g$ (composition is NOT commutative)

Example: Non-Commutativity

Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \cos x$ and $g : \mathbb{R} \to \mathbb{R}$, $g(x) = 3x^2$

$$(g \circ f)(x) = g(f(x)) = g(\cos x) = 3\cos^2 x$$

$$(f \circ g)(x) = f(g(x)) = f(3x^2) = \cos(3x^2)$$

For $x = 0$: $g \circ f(0) = 3$ but $f \circ g(0) = \cos 0 = 1$

Therefore: $g \circ f \neq f \circ g$ ✗

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9. Invertible Functions

Definition 9: Invertible Function

Function $f : X \to Y$ is **invertible** if there exists function $g : Y \to X$ such that:

$$(g \circ f)(x) = x \text{ for all } x \in X \quad (\text{identity on X})$$

$$(f \circ g)(y) = y \text{ for all } y \in Y \quad (\text{identity on Y})$$

The function g is called the **inverse of f**, denoted f^{-1}.

Critical Theorem

A function $f : X \to Y$ is invertible if and only if f is bijective (one-one and onto)

Steps to Find Inverse

1. **Verify f is bijective** (one-one and onto)
2. **Set** $y = f(x)$
3. **Solve for** x in terms of y
4. **Write** $f^{-1}(y) = $ (expression in terms of y)
5. **Verify:** $f(f^{-1}(y)) = y$ and $f^{-1}(f(x)) = x$

Example: Finding an Inverse

Let $f : \mathbb{N} \to Y$, $f(x) = 4x + 3$ where $Y = \{y \in \mathbb{N} : y = 4x + 3 \text{ for some } x \in \mathbb{N}\}$

Step 1: Check bijectivity

- One-one: $f(x_1) = f(x_2) \Rightarrow 4x_1 + 3 = 4x_2 + 3 \Rightarrow x_1 = x_2$ ✓
- Onto: By definition of Y ✓

Step 2-3: Find inverse

- $y = 4x + 3$
- $y - 3 = 4x$
- $x = \frac{y-3}{4}$

Step 4: Write inverse

$$f^{-1}(y) = \frac{y - 3}{4}$$

Step 5: Verify

- $f(f^{-1}(y)) = f\left(\frac{y-3}{4}\right) = 4 \cdot \frac{y-3}{4} + 3 = y$ ✓
- $f^{-1}(f(x)) = f^{-1}(4x + 3) = \frac{(4x+3)-3}{4} = x$ ✓

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10. Important Theorems

Theorem: Properties of Finite Sets

For a **finite set** X:

- A function $f : X \to X$ is one-one if and only if it is onto
- An onto function from X to itself is necessarily one-one
- A one-one function from X to itself is necessarily onto

Contrast: For **infinite sets**, a function can be one-one without being onto (e.g., $f(x) = 2x$ on $\mathbb{N}$)

Theorem: Intersection of Equivalence Relations

If R_1 and R_2 are equivalence relations on set A, then $R_1 \cap R_2$ is also an equivalence relation.

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11. Common Mistakes to Avoid

Mistake	Correct Approach
Confusing domain with codomain	Domain: input set; Codomain: target set
Thinking $(g \circ f)(x) = (f \circ g)(x)$	Composition is NOT commutative
Assuming all functions are invertible	Only bijective functions are invertible
Writing $f^{-1}(x)$ for non-bijective f	Inverse exists ONLY for bijective functions
Mixing up one-one and onto	One-one: no repeats; Onto: all codomain elements are images
Forgetting to verify inverse	Always verify both $f(f^{-1}(y)) = y$ and $f^{-1}(f(x)) = x$
Assuming equivalence class partitions work for any relation	ONLY true for equivalence relations

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12. Summary Table: Function Properties

Function Type	Injective	Surjective	Invertible
One-one only	✓	✗	✗
Onto only	✗	✓	✗
Bijective	✓	✓	✓
Neither	✗	✗	✗

Table 2: Function Classification

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13. Exam Checklist

- ✓ Relation = any subset of $A \times B$
- ✓ Reflexive: $(a, a) \in R$ for all $a \in A$
- ✓ Symmetric: $(a, b) \in R \Rightarrow (b, a) \in R$
- ✓ Transitive: $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$
- ✓ Equivalence relation = reflexive + symmetric + transitive
- ✓ Equivalence classes partition the set
- ✓ One-one: $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$
- ✓ Onto: Every element in codomain is an image
- ✓ Bijective = one-one AND onto = invertible
- ✓ $(g \circ f)(x) = g(f(x))$ — f applies first
- ✓ $g \circ f \neq f \circ g$ in general
- ✓ f invertible $\Longleftrightarrow$ f is bijective
- ✓ Always verify inverse by checking both compositions
- ✓ For finite set: one-one on $X \to X$ iff onto
- ✓ Intersection of equivalence relations is also equivalence

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14. Important Definitions

Relation: A subset of $A \times B$; represents connections between elements

Reflexive Relation: Each element relates to itself; $(a, a) \in R$ for all a

Symmetric Relation: If a relates to b, then b relates to a

Transitive Relation: If a relates to b and b relates to c, then a relates to c

Equivalence Relation: A relation that is reflexive, symmetric, and transitive

Equivalence Class: Set of all elements related to a given element

One-One Function: Different inputs map to different outputs (injective)

Onto Function: All codomain elements are images of some domain element (surjective)

Bijective Function: One-one and onto simultaneously (perfect correspondence)

Composition: $(g \circ f)(x) = g(f(x))$ — applying f then g

Inverse Function: f^{-1} such that $f(f^{-1}(y)) = y$ and $f^{-1}(f(x)) = x$

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15. Historical Note

The concept of function evolved from René Descartes (1596-1650), who used "function" for algebraic operations on variables. Later, Leonhard Euler (1707-1783) popularized function notation $f(x)$. The rigorous set-theoretic definition we use today comes from Lejeunne Dirichlet (1805-1859) and was formalized using modern set theory by Georg Cantor (1845-1918).

References

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[5] Cantor, G. (1874). On a property of the collection of all real algebraic numbers (Set theory foundations).